Linear Algebra Assignment

The goal of this assignment is to familiarize yourself with basics of linear algebra. Laforest's Chapter 2 is a good reference on this topic. We will not ask you to read it in detail, but going over it would certainly improve your understanding.

1 Matrix Operations

Reference: Laforest Sections 2.2,2.3

As we will see later on, quantum states are represented by vectors and operations on quantum states by matrices. Not all matrices represent valid quantum operations, only *unitary* matrices. The goal of this problem is to review matrix multiplications and the definition of a unitary matrix.

Say we are given vectors $a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and matrices $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. Those two matrices represent useful operations on quantum states, called the *Hadamard gate* and *Pauli Y-gate*.

We are asking you to evaluate the following:

- Ha, Yb
- HY
- transpose of H, Y
- complex conjugates of H, Y
- Hermitian conjugates $H^\dagger$, $Y^\dagger$, defined by taking the transpose followed by complex conjugation
- Verify that H, Y are unitary by checking that $H^\dagger H = Y^\dagger Y = I$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called *identity matrix.*
- Find the inner product of vectors a and b
- (**Extra Credit**) Show that unitary matrices preserve inner products

2 Linear Basis (Extra Credit)

Reference: Laforest Sections 2.4, 2.5, 2.6

2.1 Part a

Give an example of:

- A linear space (also called *vector space*)
- A set of linearly dependent vectors living in this space
- A set of linearly independent vectors living in this space
- A basis for this vector space

2.2 Part b

The space of 2-component vectors with real entries is denoted as R^2 and represented by a plane. Which of the following sets of vectors in R^2 form a basis?

- $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\2\end{pmatrix}\right\}$
- $\left\{\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\-1\end{pmatrix}\right\}$
- $\left\{\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\}$
- $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\}$

2.3 Part c

It is not too difficult to check that a set of vectors $\vec{u} = \begin{pmatrix}1\\1\end{pmatrix}$, $v = \begin{pmatrix}1\\-1\end{pmatrix}$ is linearly independent and forms a basis for R^2. Write the elements of the "canonical basis" $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$ as linear combinations of u and v. Find the projections of e_1 and e_2 onto u and v.

2.4 Part d

For what value of x will the following set of vectors **not** form a basis for R^3? For those special values of x, which elements of R^3 can not be written as linear combinations of the 3 vectors?

$$\{\begin{pmatrix}1\\0\\x\end{pmatrix},\begin{pmatrix}0\\1\\x\end{pmatrix},\begin{pmatrix}1\\x\\0\end{pmatrix}\} \tag{1}$$

2.5 Part e

Is a union of linear subspaces necessarily a linear subspace itself? Prove that it is or think of a counterexample.

2.6 Part f

Let V be a subspace of R^m. Suppose that $S = \{v_1, v_2, \ldots, v_n\}$ is a *spanning set* for V; that is, every vector in V can be written as a linear combination of elements of S. Prove that any set of $n + 1$ or more vectors in V is linearly dependent.